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LETTER TO THE EDITOR

Correlation equalities and coupling constant bounds implying area decay of Wilson loop for Z_2 lattice gauge theories†

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Abstract. We obtain correlation equalities for Z_2 lattice gauge theories and apply them to obtain area decay of the Wilson loop observable in a range of the coupling parameter larger than that obtained from mean field theory considerations.

In this letter we consider the well known Wilson loop observable of a Z_2 pure gauge Z^d lattice theory with Wilson action, i.e.

$$\langle W(C)\rangle = \lim_{\Lambda\to Z^d} \langle W(C)\rangle_\Lambda, \qquad \Lambda \subset Z^d,$$

where $\langle W(C)\rangle_\Lambda$ is the finite lattice Gibbs ensemble average with Wilson action Boltzmann factor $\exp(\beta\Sigma_{P\subset\Lambda}\chi_P)$ (Wilson 1974, Seiler 1982, Kogut 1979). P denotes the unit squares (plaquettes) of Λ. We let S_b denote the bond variables which take values ± 1. $W(C)$ is the product of S_b along the perimeter of the planar rectangle C of area A. $0<\beta<\infty$ is the gauge coupling constant.

Area decay of $\langle W(C)\rangle$ is a criterion for confinement. By expansion methods, it is known that area decay occurs for small β and for sufficiently large β perimeter decay occurs for $d\geqslant 3$ (Seiler 1982, Kogut 1979). For $d=2$, $\langle W(C)\rangle$ has area decay for all β by explicit calculation (Kogut 1979). We take free boundary conditions and note that Griffiths' first and second inequalities apply and therefore imply the existence of the thermodynamic limit (Glimm and Jaffe 1981).

For $d=2,3,4$ we obtain lower bounds β_L on the area decay of $\langle W(C)\rangle$, i.e. for all $\beta<\beta_L$, $\langle W(C)\rangle$ has area decay, using correlation identities and Griffiths' inequalities. The correlation identities are a gauge version of Callen identities employed by the authors (Sá Barreto and O'Carroll 1983) to obtain lower than mean field upper bounds on the critical temperature for Ising spin systems. For completeness we give a mean field type lower bound β_M ($\beta_M<\beta_L$) using a decoupling and Griffiths inequality argument (Sá Barreto and O'Carroll 1983, Tomboulis *et al* 1981).

Theorem 1. For each $\beta\subset(0,[2(d-1)]^{-1})$, $W(C)\leqslant\exp[-|\ln(2\beta(d-1))|A]$.

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Proof. For definiteness assume C lies in the x_1x_2 plane. Fix a bond b in the lower left-hand corner of C. Replace β by $\beta\lambda$, $\lambda \in [0, 1]$ in the action for the $2(d-1)$ plaquettes (call them $P_1 \ldots P_{2(d-1)}$) that have one bond in common with b. Denote the corresponding expectation by $\langle W(C)\rangle_\lambda$. Integrating $(\mathrm{d}/\mathrm{d}\lambda)\langle W(C)\rangle_\lambda$ gives, noting that $\langle W(C)\rangle_0 = 0$,

$$\begin{aligned}\langle W(C)\rangle &= \int_0^1 \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle W(C)\rangle_\lambda \\ &= \beta \int_0^1 \mathrm{d}\lambda \sum_{i=1}^{2(d-1)} (\langle W(C)\chi_{P_i}\rangle_\lambda - \langle W(C)\rangle_\lambda \langle \chi_{P_i}\rangle_\lambda) \\ &\leqslant \beta \int_0^1 \mathrm{d}\lambda \sum_{i=1}^{2(d-1)} \langle W(C)\chi_{P_i}\rangle_\lambda \leqslant \beta \sum_{i=1}^{2(d-1)} \langle W(C)\chi_{P_i}\rangle\end{aligned}$$

using Griffiths' first (second) inequality in the first (second) equality. Each term on the right corresponds to a modified contour determined by the bonds of the variables of $W(C)\chi_{P_i}$ which enlarges or diminishes C by one plaquette. We repeat the argument proceeding along successive rows of plaquettes enclosed by C. After A applications we arrive at

$$W(C) \leqslant \beta^A \text{ (sum of } [2(d-1)]^A \text{ terms).}$$

Each term is non-negative and bounded above by 1 giving $\langle W(C)\rangle \leqslant [\beta 2(d-1)]^A$.

We now give some correlation identities for $d = 2,3,4$ which are derived in a manner completely analogous to the ones for Ising spin systems (Sá Barreto and O'Carroll 1983) and will be used in theorem 3 to extend the β region of area decay given by theorem 1 and $d = 3,4$.

Theorem 2. Let $S_\mathrm{D} = S_{i_1} \ldots S_{i_\mathrm{D}}$ denote a product of distinct bond variables and for a fixed bond b occurring in S_D give a numerical ordering $1, 2, \ldots$ to the $2(d-1)$ plaquettes that have one bond in common with b. Then for:

(a) $d = 2$
$$\langle S_\mathrm{D}\rangle = a_2 \sum_i \langle S_\mathrm{D}\chi_{P_i}\rangle,$$
$$a_2 = \tfrac{1}{2}\tanh 2\beta, \qquad a_2 \geqslant 0;$$

(b) $d = 3$:
$$\langle S_\mathrm{D}\rangle = a_3 \sum_i \langle S_\mathrm{D}\chi_{P_i}\rangle + b_3 \sum_{i<j<k} \langle S_\mathrm{D}\chi_{P_i}\chi_{P_j}\chi_{P_k}\rangle,$$
$$a_3 = 2^{-3}(\tanh 4\beta + 2\tanh 2\beta), \qquad a_3 \geqslant 0,$$
$$b_3 = 2^{-3}(\tanh 4\beta - 2\tanh 2\beta), \qquad b_3 \leqslant 0;$$

(c) $d = 4$:
$$\langle S_\mathrm{D}\rangle = a_4 \sum_i \langle S_\mathrm{D}\chi_{P_i}\rangle + b_4 \sum_{i<j<k} \langle S_\mathrm{D}\chi_{P_i}\chi_{P_j}\chi_{P_k}\rangle + c_4 \sum_{i<j<k<l<m} \langle S_\mathrm{D}\chi_{P_i}\chi_{P_j}\chi_{P_k}\chi_{P_l}\chi_{P_m}\rangle,$$
$$a_4 = 2^{-5}(\tanh 6\beta + 4\tanh 4\beta + 5\tanh 2\beta), \qquad a_4 \geqslant 0,$$
$$b_4 = 2^{-5}(\tanh 6\beta - 3\tanh 2\beta), \qquad b_4 \leqslant 0,$$
$$c_4 = 2^{-5}(\tanh 6\beta - 4\tanh 4\beta + 5\tanh 2\beta), \qquad c_4 \geqslant 0.$$

Proof. We have

$$\langle S_D\rangle = \frac{1}{Z}\sum_{\{S\}} S_D \exp\Big(\beta \sum_{P\subset\Lambda} \chi_P\Big),$$

where

$$Z = \sum_{\{S\}} \exp\Big(\beta \sum_{P\subset\Lambda} \chi_P\Big).$$

Let us consider the bond b, with S_b, and the plaquette χ_b which contains the bond b. Let $S_D^{(b)}$ be the product S_D with the bond b deleted. We have

$$\langle S_D\rangle = \frac{1}{Z}\Big[\sum_{\{S\}} S_D^{(b)}\Big(\frac{\Sigma_{S_b} S_b \exp(\beta\chi_b)}{\Sigma_{S_b}\exp(\beta\chi_b)}\Big)\exp\Big(\beta\sum_{P\subset\Lambda}\chi_P\Big)\Big].$$

Summing over S_b and introducing $D \equiv \partial/\partial x$ through $e^{\alpha D} f(x) = f(x+\alpha)$, we get

$$\langle S_D\rangle = \Big\langle S_D^{(b)} \tanh\beta \sum_{(\text{neighbours of } b)} S_k S_l S_m\Big\rangle$$

$$= \Big\langle S_D^{(b)} \prod_{(\text{neighbours of } b)} [\cosh\beta D + (S_k S_l S_m)\sinh\beta D]\Big\rangle \tanh x\Big|_{x=0}.$$

Applying this result for $d = 2,3,4$ gives, after some algebra, (a), (b) and (c).

Remark 1. On the right-hand side of these equations note that S_b is absent since it always appears an even number of times in each term.

Remark 2. Other equalities, such as Euclidean lattice equations of motion for averages of gauge invariant observables, could also be obtained as previously (Sá Barreto and O'Carroll 1983).

Theorem 3. Let β be such that:

(a) $4a_3 < 1$. Then $\langle W(C)\rangle \leq \exp(-|\ln[4a_3]|A)$ for $d = 3$.

(b) $6(a_4 + c_4) < 1$. Then $\langle W(C)\rangle \leq \exp(-|\ln[6(a_4+c_4)]|A)$ for $d = 4$.

Proof. (a) We use the same method as in the proof of theorem 1 except we use theorem 2(b). For $b \in C$

$$\langle W(C)\rangle = a_3 \sum_i \langle W(C)\chi_{P_i}\rangle + b_3 \sum_{i<j<k} \langle W(C)\chi_{P_i}\chi_{P_j}\chi_{P_k}\rangle$$

$$\leq a_3 \sum_i \langle W(C)\chi_{P_i}\rangle$$

since b_3 is negative and $\langle W(C)\chi_{P_i}\chi_{P_j}\chi_{P_k}\rangle$ is positive by Griffiths' first inequality. At each application of the equality we pass to an inequality by dropping the b_3 terms; after A steps we arrive at

$$\langle W(C)\rangle \leq a_3^A \text{ (sum of } 4^A \text{ terms)}$$

where each term is less than one.

(b) As in (a) but using theorem 2(c). We can drop the b_4 terms in favour of an inequality. At each stage we have six terms from the a_4 term and six terms from the c_4 term.

References

Glimm J and Jaffe A 1981 *Quantum Physics* (New York: Springer)
Kogut J 1979 *Rev. Mod. Phys.* **51** 659
Sá Barreto F C and O'Carroll M 1983 *J. Phys. A: Math. Gen.* **16** 1035–9
Seiler E 1982 *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics, Lect. Notes in Physics* 159
Tomboulis E, Ukawa A and Windey P 1981 *Nucl. Phys.* B **180** [FS2] 294–300
Wilson K 1974 *Phys. Rev.* D **10** 2445–59